CONCENTRATION OF MEASURE VIA APPROXIMATED BRUNN–MINKOWSKI INEQUALITIES

MASAYOSHI WATANABE

ABSTRACT. We prove that an approximated version of the Brunn–Minkowski inequality with volume distortion coefficient implies a Gaussian concentration-of-measure phenomenon. Our main theorem is applicable to discrete spaces.

1. INTRODUCTION

Let (X, d) be a complete separable metric space equipped with a Borel probability measure μ on X with full support. Henceforth, we call such a triple a *metric measure space*. The *concentration function* of a metric measure space (X, d, μ) is defined by

$$\alpha_{(X,\,d,\,\mu)}(r) = \sup\{1 - \mu(A_r) \,|\, A \text{ is a Borel set in } X \text{ with } \mu(A) \geq 1/2\},$$

where $A_r := \{x \in X \,|\, d(x, A) < r\}$.

Let (M, g) be an n-dimensional complete Riemannian manifold with Riemannian distance d_g and normalized Riemannian measure μ_g. If the Ricci curvature of M is bounded below by $n-1$, then the Lévy–Gromov isoperimetric inequality [3, Appendix C] implies

$$\alpha_{(M,\,d_g,\,\mu_g)}(r) \leq e^{-(n-1)r^2/2}$$

for every $r > 0$. This is an example of *Gaussian concentration-of-measure phenomenon*. See [4] for details. Moreover, the curvature-dimension condition $\mathrm{CD}(n-1, n)$, or n-Ricci curvature $\geq n-1$, for measured length spaces implies a Gaussian concentration via log-Sobolev inequality; see [5, Corollary 6.12], [6], and [4, Theorem 5.3].

In this paper we deduce a Gaussian concentration from a weaker condition: an ϵ-*approximated Brunn–Minkowski inequality* ϵ-$\mathrm{BM}(n-1, n)$ *of dimension* n *and of Ricci curvature* $\geq n-1$, introduced by Bonnefont [1]. The definition of ϵ-$\mathrm{BM}(n-1, n)$ is in Section 2. Our main theorem is

Theorem 1.1. *Let* $\epsilon \geq 0$ *and* $n \in (1, \infty)$. *If a metric measure space* (X, d, μ) *satisfies* ϵ-$\mathrm{BM}(n-1, n)$, *then we have, for every* $r > 0$,

$$\alpha_{(X,\,d,\,\mu)}(r) \leq 2e^{-(n-1)r^2/\pi^2}.$$

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Note that the curvature-dimension condition $\mathrm{CD}(n-1,n)$ does not make sense in discrete spaces; however, ϵ-$\mathrm{BM}(n-1,n)$ does. For example, we can apply Theorem 1.1 to the discretization of a measured length space: let us explain what we mean by the discretization in Section 4.

Corollary 1.2. *Given $\epsilon \geq 0$ and $n \in (1,\infty)$, let $(X_\epsilon, d, \mu_\epsilon)$ be a discretization of a measured length space with the curvature-dimension condition* $\mathrm{CD}(n-1,n)$. *Then we have, for every $r > 0$,*

$$\alpha_{(X_\epsilon,\, d,\, \mu_\epsilon)}(r) \leq 2e^{-(n-1)r^2/\pi^2}.$$

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2. Approximated Brunn–Minkowski inequality

Let (X,d,μ) be a metric measure space. Given $\epsilon \geq 0$, $t \in (0,1)$, and $A_0, A_1 \subset X$, we first define the set of *ϵ-approximated t-intermediate points* between A_0 and A_1 by

$$\begin{aligned} I_t^\epsilon(A_0,A_1) = \big\{ x \in X \;\big|\; &\text{there exist } x_0 \in A_0 \text{ and } x_1 \in A_1 \text{ with} \\ &|d(x_0,x) - td(x_0,x_1)| \leq \epsilon \text{ and } |d(x,x_1) - (1-t)d(x_0,x_1)| \leq \epsilon \big\}. \end{aligned}$$

In Euclidean space, $I_t^0(A_0,A_1)$ coincides with the Minkowski sum $(1-t)A_0 + tA_1$.

Definition 2.1. Let $\epsilon \geq 0$ and $n \in (1,\infty)$. We say that (X,d,μ) satisfies an *ϵ-approximated Brunn–Minkowski inequality of dimension n and of Ricci curvature $\geq n-1$* or, for short, ϵ-$\mathrm{BM}(n-1,n)$ if we have

$$\begin{aligned} \mu(I_t^\epsilon(A_0,A_1))^{1/n} \geq (1-t) &\left[\inf_{x_0 \in A_0,\, x_1 \in A_1} \left(\frac{\sin((1-t)d(x_0,x_1))}{(1-t)\sin d(x_0,x_1)} \right)^{(n-1)/n} \right] \mu(A_0)^{1/n} \\ &+ t \left[\inf_{x_0 \in A_0,\, x_1 \in A_1} \left(\frac{\sin(td(x_0,x_1))}{t \sin d(x_0,x_1)} \right)^{(n-1)/n} \right] \mu(A_1)^{1/n} \end{aligned} \tag{2.1}$$

for all nonempty Borel sets $A_0, A_1 \subset X$ and for all $t \in (0,1)$, where

$$\frac{\sin(td(x_0,x_1))}{t\sin d(x_0,x_1)} := +\infty \qquad \text{if } \; d(x_0,x_1) \geq \pi.$$

See [9, Section 14] for the meaning of distortion coefficients in (2.1). Clearly, ϵ-$\mathrm{BM}(n-1,n)$ implies ϵ'-$\mathrm{BM}(n-1,n)$ for $\epsilon' \geq \epsilon$. The curvature-dimension condition $\mathrm{CD}(n-1,n)$ implies 0-$\mathrm{BM}(n'-1,n')$ for all $n' \geq n$; see [8, Proposition 2.1] and [9, Theorem 30.7]. The Brunn–Minkowski inequality in curved spaces is proved by virtue of [2].

3. Concentration of measure

We begin with a lemma corresponding to the Bonnet–Myers theorem; see [8, Corollary 2.6] and [9, Proposition 29.11].

Lemma 3.1. *Let $\epsilon \geq 0$ and $n \in (1,\infty)$. If a metric measure space (X,d,μ) satisfies ϵ-BM$(n-1,n)$, then* $\operatorname{diam}(X) \leq \pi$.

Proof. Suppose that there are two points $x_0, x_1 \in X$ with $d(x_0,x_1) > \pi$. Choosing a sufficiently small $\delta > 0$, we have $d(B_\delta(x_0), B_\delta(x_1)) > \pi$. Note that, in (2.1) with $A_0 = B_\delta(x_0)$ and $A_1 = B_\delta(x_1)$, the coefficients in the right-hand side equals $+\infty$. We then have a contradiction from $\mu(A_0) > 0$, $\mu(A_1) > 0$, and $\mu(I_t^\epsilon(A_0,A_1)) \leq \mu(X) = 1$. □

Proof of Theorem 1.1. Let A be a Borel set in X with $\mu(A) \geq 1/2$. By Lemma 3.1, it suffices to prove $1-\mu(A_r) \leq 2e^{-(n-1)r^2/\pi^2}$ for every $r \in (0,\pi)$.

We now put $B = X \setminus A_r$ for a fixed $r \in (0,\pi)$. Note that $(\sin(d/2))/((1/2)\sin d)$ is monotone nonincreasing in $d \in (0,\pi)$. Since $d(A,B) \geq r$, it follows that

$$\inf_{x\in A,\, y\in B}\left(\frac{\sin((1/2)d(x,y))}{(1/2)\sin d(x,y)}\right)^{(n-1)/n} \geq \left(\frac{\sin(r/2)}{(1/2)\sin r}\right)^{(n-1)/n}.$$

Inequality (2.1) with $A_0 = A$, $A_1 = B$, and $t = 1/2$ gives

$$\begin{aligned}\mu(Z_{1/2}^\epsilon(A,B))^{1/n} &\geq \frac{1}{2}\left(\frac{\sin(r/2)}{(1/2)\sin r}\right)^{(n-1)/n}(\mu(A)^{1/n}+\mu(B)^{1/n}) \\ &\geq \frac{(\mu(A)^{1/n}\mu(B)^{1/n})^{1/2}}{(\cos(r/2))^{(n-1)/n}}.\end{aligned}$$

We used relations $\sin r = 2\sin(r/2)\cos(r/2)$ and $(a+b)/2 \geq \sqrt{ab}$ for the last step. Noting $\mu(A) \geq 1/2$ and $\mu(Z_{1/2}^\epsilon(A,B)) \leq \mu(X) = 1$, we get

$$\mu(B) \leq 2\Big(\cos\frac{r}{2}\Big)^{2(n-1)} \leq 2\Big(1-\frac{r^2}{2\pi^2}\Big)^{2(n-1)} \leq 2e^{-(n-1)r^2/\pi^2}.$$

□

We can get a better estimate for all sufficiently large $n \in (1,\infty)$ and small $r > 0$.

Theorem 3.2. *Fix $\epsilon \geq 0$. Given $\delta > 0$, there exist $n_0 \in (1,\infty)$ and $r_0 > 0$ such that if a metric measure space (X,d,μ) satisfies ϵ-BM$(n-1,n)$ for a number $n \geq n_0$, then we have, for $0 < r \leq r_0$,*

$$\alpha_{(X,d,\mu)}(r) \leq e^{-(1-\delta)nr^2/4}.$$

Proof. Modify the proof of Theorem 1.1. We get

$$\alpha_{(X,d,\mu)}(r) \leq e^{-n[1+2^{-1/n}-2(\cos(r/2))^{(n-1)/n}]}$$

without employing the arithmetic-geometric mean inequality. Taylor expansion, $2 - 2\cos(r/2) = r^2/4 + o(r^2)$, completes the proof. □

4. Discretization

Let (X, d, μ) be a metric measure space. Given $\epsilon > 0$, take a set $\{x_i\}_{i=1}^{\infty}$ of countable distinct points in X with $X = \bigcup_{i=1}^{\infty} B_\epsilon(x_i)$, where $B_\epsilon(x_i)$ is the open ball of radius ϵ centered at x_i. We can choose a measurable set $A_i \subset B_\epsilon(x_i)$ for each i such that $x_i \in A_i$, $A_i \cap A_j \neq \emptyset$ $(i \neq j)$, and $X = \bigcup_{i=1}^{\infty} A_i$. Setting $\mu_\epsilon(\{x_i\}) = \mu(A_i)$, we get a probability measure μ_ϵ on $X_\epsilon := \{x_i\}_{i=1}^{\infty}$. We call $(X_\epsilon, d, \mu_\epsilon)$ a *discretization* of (X, d, μ).

Proof of Corollary 1.2. Every discretization $(X_\epsilon, d, \mu_\epsilon)$ of a measured length space (X, d, μ) with $\mathrm{CD}(n-1, n)$ satisfies 4ϵ-$\mathrm{BM}(n-1, n)$ [1, Section 3]; therefore, Theorem 1.1 completes the proof. $\square$

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Mathematical Institute, Tohoku University, Sendai, 980-8578, Japan
E-mail address: sa3m33@math.tohoku.ac.jp